Rate-optimal gamma scale mixture detection

Joint work with Qikun Chen

Michael Stewart

Biometrics in the Bush Capital
Canberra
26th November 2025

School of Mathematics and Statistics
The University of Sydney

THE UNIVERSITY OF SYDNEY

Outline

Motivating example: ion channel openings

Gamma scale mixture detection

The case $\mathbf{0} < \theta < \mathbf{1}$ for general α.

Motivating example: ion channel openings

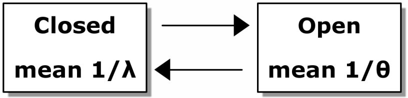
Modelling ion channels opening times

- Neurotransmitter is released across the synapse via the opening and closing of calcium ion channels.
- Opening times can be measured.
- One scientific question has been:

Is there a single open state or multiple open states?

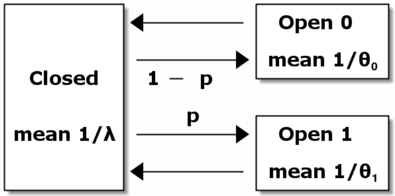
Continuous-time Markov chains

- Times in different states have been modelled using continuous-time Markov chains.
  - Times in each state are exponentially distributed.
- If only a single "open" state, a series of measurements should resemble a sample from an exponential distribution.
- Each observed opening time X_i satisfies $P(X_i > x) = e^{-\theta x}$.



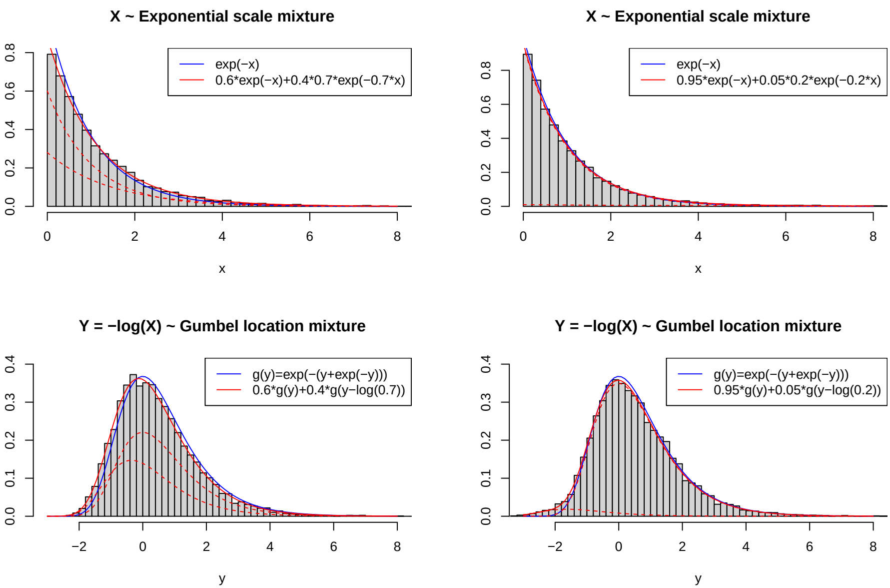
Multiple states give a mixture

- If there are two (different) *open* states but the measuring device cannot distinguish between them, opening times form a **mixture** of two exponential samples.
- Each observed opening time X_i satisfies $P(X_i > x) = (1-p)e^{-\theta_0 x} + pe^{-\theta_1 x}$.



Hard-to-detect local alternatives

- Choosing between one or two open states can be formulated as a hypothesis-testing problem.
- There are two ways a “local alternative” can be hard to detect:
  1. the two means are close to each other
  2. the mixing proportion p is close to 0 (or 1)
- These two have quite different behaviour, case 2. being the “most challenging”.


X ~ Exponential scale mixture
exp(−x)
0.6*exp(−x)+0.4*0.7*exp(−0.7*x)
x
X ~ Exponential scale mixture
exp(−x)
0.95*exp(−x)+0.05*0.2*exp(−0.2*x)
x
Y = −log(X) ~ Gumbel location mixture
g(y)=exp(−(y+exp(−y)))
0.6*g(y)+0.4*g(y−log(0.7))
y
Y = −log(X) ~ Gumbel location mixture
g(y)=exp(−(y+exp(−y)))
0.95*g(y)+0.05*g(y−log(0.2))
y

Gamma scale mixture detection

Hypothesis testing problem of interest

- We model data as iid random varibles $X_1, \ldots, X_n$.
- Let $F_\alpha(\cdot)$ denote the gamma$(\alpha, 1)$ CDF (α is **known**).
- We are interested in the hypothesis testing problem

$$H_0\colon P(X_1 \le x) = F_\alpha(x) \;\text{ vs. }\; H_1\colon P(X_1 \le x) = (1-p)F_\alpha(x) + pF_\alpha(\theta x)\,,$$

  for $0 < p < 1$ *and* $\theta \neq 1$.
- This is the simplest possible "gamma scale mixture detection" model.

Distinguishability

- For a given sequence (p_n, θ_n), write $G_n(x) = (1 - p_n)F_\alpha(x) + p_n F_\alpha(\theta_n x)$.
- We address the question:

  How close to $F_\alpha(\cdot)$ can the mixture $G_n(\cdot)$ be and still be "detectable"?

  That is, is there a test so that under $G_n(\cdot)$ we have power $\to 1$?
- We focus on the "**sparse mixture**" case where $p_n \to 0$.

Previous work

- Deep general results for exponential families provided by Ditzhaus (2019) covered the gamma scale mixture where $\theta > 1$;
  - this is when the contaminating mean is **smaller** than the null.
- Arias-Castro and Huang (2020) covered the case $0 < \theta < 1$ for $\alpha = \frac{1}{2}$;
  - this is a χ_1^2 scale mixture;
  - applicable if we have a normal mixture variance with common known mean.

The case $0 < \theta < 1$ for general α.

Separation in *n* dimensions

- In one sense the problem is "easy": if
  - F_α^n and G_n^n are the n-dimensional versions of H_0 and H_1;
  - if the **total variation** (TV, i.e. half L_1) distance

$$d_{\mathrm{TV}}(F_\alpha^n, G_n^n) = \sup_A |F_\alpha^n\{A\} - G_n^n\{A\}| \to \begin{cases} 1 & \text{then NP test has limiting power 1;} \\ 0 & \text{then NP test has no limiting power.} \end{cases}$$

- The difficulty is in approximating TV distance in n dimensions.

Hellinger distance trick

- The Hellinger distance whose square is

$$d_H^2(F_\alpha^n, G_n^n) = \int \left(\sqrt{dF_\alpha^n} - \sqrt{dG_n^n}\right)^2 = 2\left[1 - \int \sqrt{dF_\alpha^n dG_n^n}\right]$$

  $\to 0$ if and only if $d_{\mathrm{TV}}(F_\alpha^n, G_n^n) \to 0$.

- Since $\int \sqrt{dF_\alpha^n dG_n^n} = \left(\int \sqrt{dF_\alpha dG_n}\right)^n$ there is a nice relationship between $d_H(F_\alpha, G_n)$ and $d_H(F_\alpha^n, G_n^n)$:

$$d_H^2(F_\alpha^n, G_n^n) = 2\left\{1 - \left[1 - \frac{1}{2} d_H^2(F_\alpha, G_n)\right]^n\right\}.$$

- So if $n d_H^2(F_\alpha, G_n) \to 0$, $d_{\mathrm{TV}}(F_\alpha^n, G_n^n) \to 0$ too.

Critical rate r_n under 4 scenarios

- In S. (2022) we showed that under each scenario for θ_n, that if $p_n = o(r_n)$, $nd_H^2(F_\alpha, G_n) \to 0$:

	Scenario	r_n
1.	$\theta_n = 1 - \Delta_n \uparrow 1$	$n^{-1/2}\Delta_n^{-1}$
2.	$\theta_n \equiv \theta \in (\frac{1}{2}, 1)$ fixed	$n^{-1/2}$
3.	$\theta_n \equiv \frac{1}{2}$ fixed	$[n(\log n)^\alpha]^{-1/2}$
4.	$\theta_n \equiv \theta \in (0, \frac{1}{2})$ fixed	$[n(\log n)^{\alpha-1}]^{\theta-1}$

▸ slide 15

Bonferroni test attains r_n in 3 scenarios

- A test based on the sample mean or median can detect $p_n \asymp r_n$ (i.e. attains critical rate) in scenarios 1 and 2.
- A test based on the sample maximum attains the critical rate in scenario 4.
- A Bonferroni test using the smallest p-value of these two attains the critical rate in scenarios 1, 2 and 4;
  - it does **not** attain the critical rate in scenario 3 ($\theta_n \equiv \frac{1}{2}$) though.

Score test attains critical rate in scenario 3

- The score statistic for testing $H_0: p = 0$ vs. $H_1: p > 0$ when $\theta \equiv \frac{1}{2}$ is known is $\sum_{i=1}^n e^{X_i/2}$.
- Note that under H_0, $Var\left(\sum_{i=1}^n e^{X_i/2}\right) = \infty$!
- Nonetheless, in Chen and S. (2024) we showed that under scenario 3,

$$\frac{\sum_{i=1}^n \left[e^{X_i/2} - 2^\alpha\right]}{\sqrt{n(\log n)^\alpha}} \xrightarrow{d} \begin{cases} N\left(0, \Gamma(\alpha+1)^{-1}\right) & \text{if } p_n \equiv 0, \\ N\left(\mu, \Gamma(\alpha+1)^{-1}\right) & \text{if } p_n \sim \dfrac{\mu 2^\alpha \Gamma(\alpha+1)}{\sqrt{n(\log n)^\alpha}} \end{cases}$$

  and thus attains the critical rate.
- Thus a Bonferroni test based on the smallest of 3 p-values (this score test plus sample mean and max) attains critical rate in all 4 scenarios.

Current/future work

- Note that the power of $\log n$ in r_n varies in a non-continuous way as $\theta \to \frac{1}{2}$.

  slide 12

- It turns out that in a scenario where $\theta_n \to \frac{1}{2}$ *slowly enough* then a different critical rate is obtained:
  - e.g. if $\theta_n = \frac{1+\Delta_n}{2} \downarrow \frac{1}{2}$, $r_n = \sqrt{\frac{\max\left(\Delta_n, \frac{1}{\log n}\right)^{\alpha}}{n}}$
  - the Bonferroni test may not attain this critical rate;
  - indeed *no* adaptive test (i.e. without knowledge of Δ_n) may be able to.
- A broader "asymptotic minimax" framework may be needed:
  - a "price for adaptivity" like the extra $\sqrt{\log\log n}$ factor seen in the analogous normal location mixture problem (see Ingster (1997, 2001, 2002)), may apply;
  - the GLRT would then be optimal according to that, and thus **not** attain the critical rates in our 4 scenarios.

THANK YOU!

References

Ery Arias-Castro and Rong Huang. The sparse variance contamination model. *Statistics*, 54(5):1081–1093, 2020. ISSN 0233-1888.

Qikun Chen and M. S. Optimal detection of sparse gamma scale admixture with twice the null mean. *Statist. Probab. Lett.*, 209:Paper No. 110083, 6, 2024. ISSN 0167-7152,1879-2103.

Marc Ditzhaus. Signal detection via phi-divergences for general mixtures. *Bernoulli*, 25(4A):3041–3068, 2019. ISSN 1350-7265.

Yu. I. Ingster. Some problems of hypothesis testing leading to infinitely divisible distributions. *Math. Methods Statist.*, 6(1):47–69, 1997. ISSN 1066-5307.

Yu. I. Ingster. Adaptive detection of a signal of growing dimension. I. *Math. Methods Statist.*, 10(4):395–421 (2002), 2001. ISSN 1066-5307. Meeting on Mathematical Statistics (Marseille, 2000).

Yu. I. Ingster. Adaptive detection of a signal of growing dimension. II. *Math. Methods Statist.*, 11(1):37–68, 2002. ISSN 1066-5307.

M. S. Detection boundary for a sparse gamma scale mixture model. *Aust. N. Z. J. Stat.*, 64(2):282–296, 2022. ISSN 1369-1473.